Faster Root Counting Over $\mathbb{Z}/(p^k)$

J. Maurice Rojas
July 19, 2018

This is joint work with...

Leann Kopp Natalie Randall Yuyu Zhu

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...and is heavily based on the joint work with Qi Cheng, Shuhong Gao, and Daqing Wan just presented here!

Main Result

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- For any fixed k, detecting roots in $\mathbb{Z}/(p^k)$ is **NP**-hard...
- Detecting roots in $\mathbb{Q}_p$ for an input $(f, p) \in \mathbb{Z}[x_1] \times \{2, 3, 5, \ldots\}$ is **NP**-hard with respect to **ZPP** reductions [Avendaño, Ibrahim, Rojas, Rusek, 2011].

Key Reduction for Recursion

We'll *count* $\zeta = \zeta_0 + p\zeta_1 + \cdots + p^{k-1}\zeta_{k-1} \in \mathbb{Z}/(p^k)$ by first *finding* possible $\zeta_0 \in \{0, \ldots, p-1\}$,

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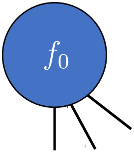

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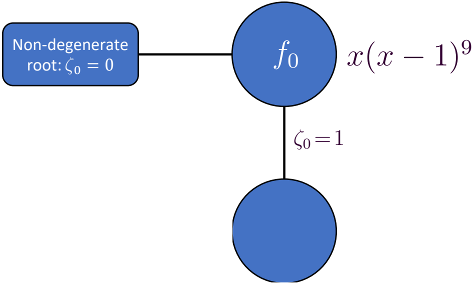

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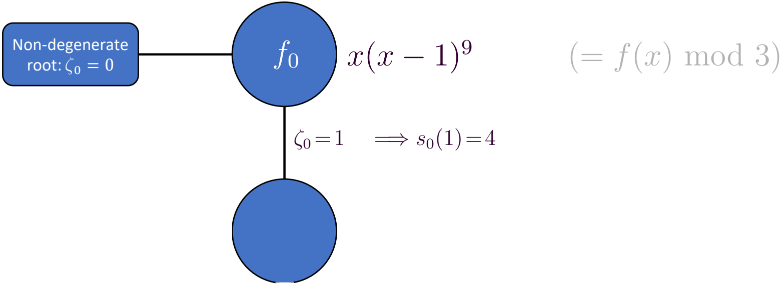

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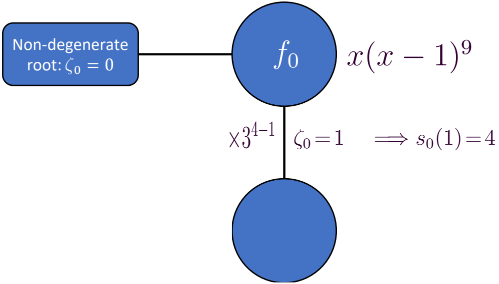

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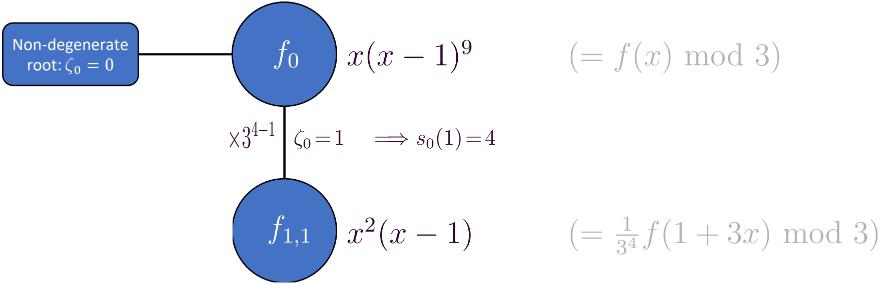

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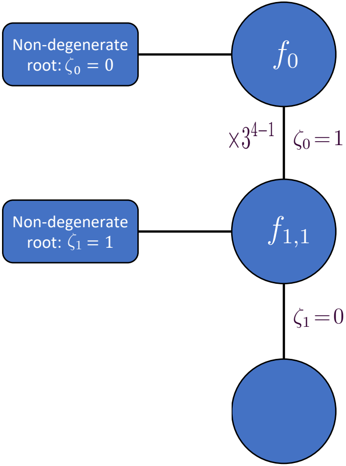

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$\Longrightarrow s_0(1) = 4$

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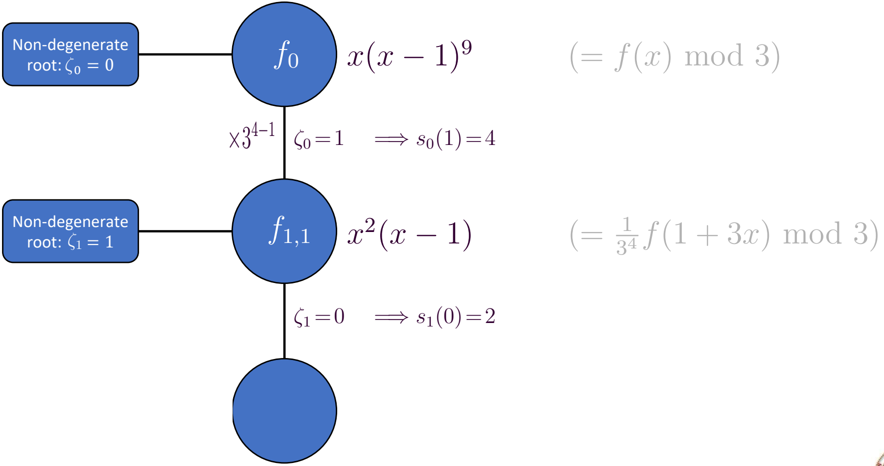

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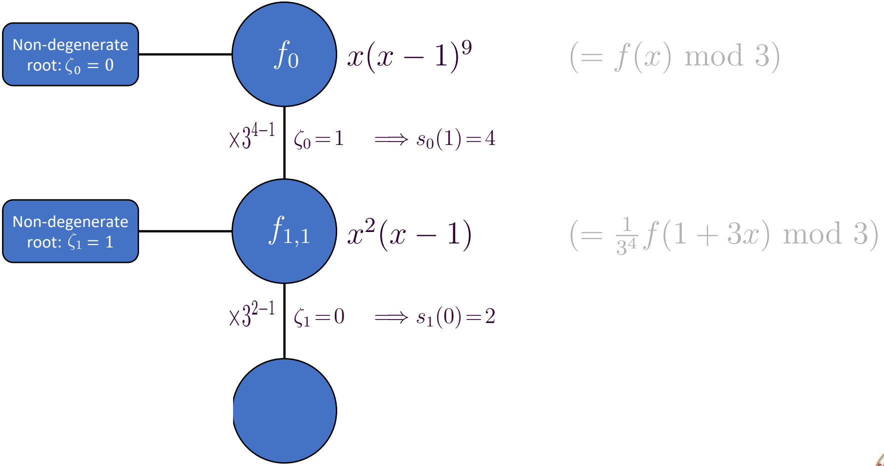

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Non-degenerate root: $\zeta_0 = 0$ — f_0: $x(x-1)^9$ $\quad (= f(x) \bmod 3)$

$\times 3^{4-1}$ $\quad \zeta_0 = 1 \implies s_0(1) = 4$

Non-degenerate root: $\zeta_1 = 1$ — $f_{1,1}$: $x^2(x-1)$ $\quad (= \frac{1}{3^4} f(1+3x) \bmod 3)$

$\times 3^{2-1}$ $\quad \zeta_1 = 0 \implies s_1(0) = 2$

$f_{2,0}$: $2(x-1)(x-2)$ $\quad (= \frac{1}{3^{4+2}} f(1 + 0 \cdot 3 + 3^2 x) \bmod 3)$

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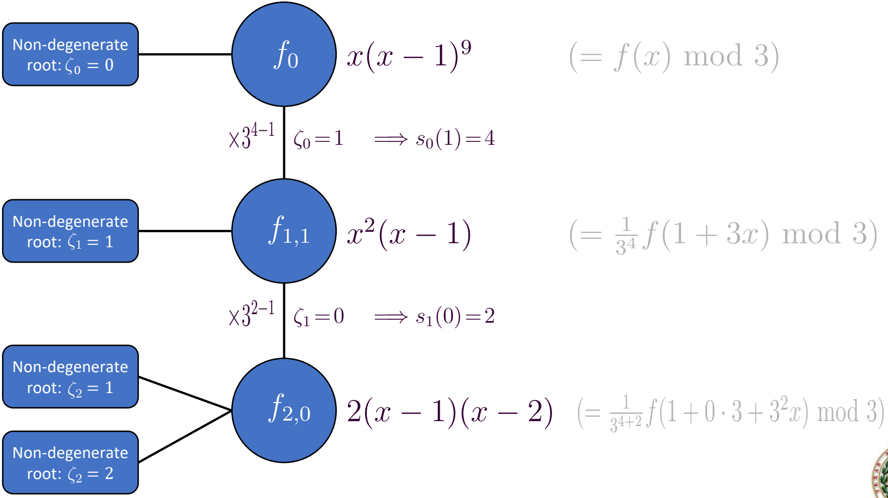

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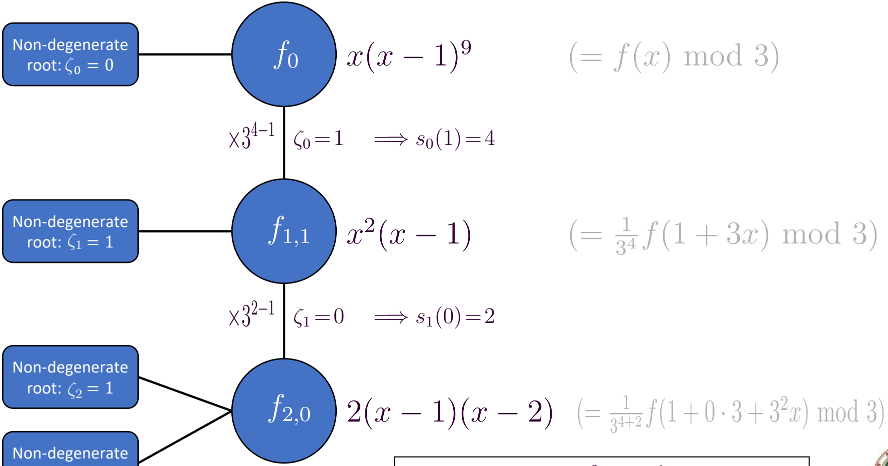

Final Count $= 1 + 3^3(1 + 3^1(1+1)) = 190.$

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♡ Thank you for your attention!

See www.math.tamu.edu/~rojas for preprints and further info...

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- Proceed recursively!