Joint work with:

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If X has genus g, and X/G has genus h, and those branch points have monodromy of order $m_1, \ldots, m_r$, respectively, then

$$[h; m_1, \ldots, m_r]$$

is the **signature** of the action of G on X.

A finite group G acts on a compact Riemann surface X of genus $g \geq 2$ with signature $[h; m_1, \ldots, m_r]$ if and only if:

I. the Riemann-Hurwitz formula is satisfied (with m_i the orders of elements in G):

$$g = 1 + |G|(h-1) + \frac{|G|}{2} \sum_{j=1}^{r} \left(1 - \frac{1}{m_j}\right),$$

II. there exists a *generating vector* $(a_1, b_1, \ldots a_h, b_h, c_1, \ldots, c_r)$ of elements of G which satisfies the following properties:

1. $G = \langle a_1, b_1, a_2, b_2, \ldots, a_h, b_h, c_1, \ldots, c_r \rangle$.
2. The order of c_j is m_j for $1 \leq j \leq r$.
3. $\prod_{i=1}^{h} [a_i, b_i] \prod_{j=1}^{r} c_j = e_G$, the identity in G.

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actual signatures satisfy I. and II.

These are not always the same.

Example 1

$[0; 3, 3, 9]$ satisfies Riemann-Hurwitz for a curve of genus 2 and a group of order 9. But this signature cannot be an actual signature for abelian groups. (There's an issue with the lcm of the m_i.) But all groups of order 9 are abelian.

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Sometimes they are badly not the same for a fixed group.

Example 2

Take $q = p^n$ for p an odd prime. Then $[0; \underbrace{2, 2, \ldots, 2}_{r>4}]$ is a potential signature for $\mathrm{SL}(2, q)$, but only one element of order 2. r copies of this element of order 2 will never generate $\mathrm{SL}(2, q)$.

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The picture for non-abelian p-groups is not so clear yet.